1.3 Vector Equations

Sunday, January 27, 2019 1:53 PM

Big idea:
- concept of vectors (in $\mathbb{R}^n$)
- algebra of vectors
- geometry of vectors

Recap: recall definitions from yesterday, with examples from handouts.
Let's go through part 2 of row reduction handout and discuss the summary
↳ notice all this stuff is now posted on the course page

Vectors (in $\mathbb{R}^n$)

"A key mathematical object in linear algebra, we will study them in a formal manner later but for now, note that they are fundamentally different object from numbers, but they are still "algebraic" in the sense that we can manipulate and combine these objects by familiar algebraic operations. As the course proceeds we will gain deep understanding of what a vector is, and why they are useful. For today, though, we start simple."

Def A (column) vector (in $\mathbb{R}^n$) is a $n \times 1$ matrix.
"that is a single column with n entries, all of which are real numbers. We will later see that many other natural objects are vectors (e.g. lists of complex numbers, forces in a physical system, continuous functions).

Ex Vectors in $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^5$

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 6 \\ \pi \\ 7 \end{bmatrix} \quad \vec{w}_1 = \begin{bmatrix} 2 \\ 3 \\ 1/2 \\ .3 \\ .2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 11 \end{bmatrix}$$

When typing we use bold face for vectors: $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

In general, any list of n real numbers, $u_1, u_2, \ldots, u_n$ gives a vector in $\mathbb{R}^n$, namely

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}.$$

Notice the hat vs. no hat.

Algebra of vectors:

Algebraic structure can be very naturally given to vectors in $\mathbb{R}^n$, we begin with equality but move to given two vectors in $\mathbb{R}^n$, we begin with equality but move to summation and scalar multip.

Equality: Two vectors $\vec{u}, \vec{v}$ are equal, $\vec{u} = \vec{v}$, if their entries are equal.

Ex $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ but $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4/2 \\ 9/3 \\ 2^2 \end{bmatrix}$

Summation: The sum of two vectors, $\vec{u}, \vec{v}$, is the vector $\vec{u} + \vec{v}$ obtained by adding corresponding entries.

Ex $\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+3 \\ 4+(-2) \\ 5+0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$

Scalar Multiplication: Given a vector $\vec{u}$ and real number c, the scalar multiple of $\vec{u}$ by c, written $c\vec{u}$, is the vector obtained by multiplying each entry of $\vec{u}$ by c.

> We distinguish the scalar c by us hat vs the vector $\vec{u}$, in typeface c is light face while **u** is bold face.

Ex $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $c = 6$ then $c\vec{u} = 6\vec{u} = 6\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6\cdot 2 \\ 6\cdot 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$

In general, $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, we have

$$c\vec{u} + \vec{v} = c\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} cu_1 + v_1 \\ \vdots \\ cu_n + v_n \end{bmatrix}$$

Using this notation we can verify the following

Algebraic properties of vectors

$\vec{u}, \vec{v}, \vec{w}$ are vectors in $\mathbb{R}^n$, c, d are scalars from $\mathbb{R}$ and $\vec{0}$ (**0** in typing) is the vector of all zeros, commonly called the zero vector.

1) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2) $(\vec{u} + \vec{v}) + \vec{w} = u + (\vec{v} + \vec{w})$
3) $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$
4) $\vec{u} + (-\vec{u}) = -\vec{u} + \vec{u} = 0$

   where $-\vec{u} = (-1)\vec{u}$
5) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
6) $(c+d)\vec{u} = c\vec{u} + d\vec{u}$
7) $c(d\vec{u}) = (cd)\vec{u}$
8) $1\vec{u} = \vec{u}$

Note: subtraction: $\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v}$

Def Given vectors $\vec{v}_1, \dots, \vec{v}_p$ in $\mathbb{R}^n$ and scalars $c_1, \dots, c_p$, the vector

$$y = c_1\vec{v}_1 + \dots + c_p\vec{v}_p$$

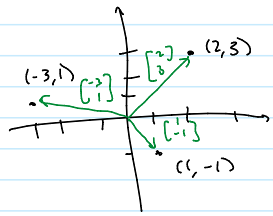
is called a linear combination of $\vec{v}_1, \dots, \vec{v}_p$ with weights $c_1, \dots, c_p$.

[?] Before or after geometry?

Geometry of Vectors

Geometry of Vectors

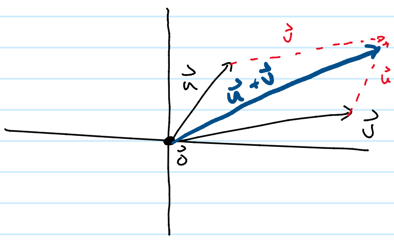
As each point in the plane is given by an ordered pair of real numbers, we identify (a,b) with the vector $\begin{bmatrix} a \\ b \end{bmatrix}$.



Alternatively, we may view a vector graphically, as the arrow pointing to the point determined by its entries from the origin, so

Thus, two vectors are equal if they have the same direction and length.

Summation and scalar multiplication:



- Summation: $\vec{u}+\vec{v}$ is the vector corresponding to the fourth vertex of the parallelogram with vertices at $\vec{0}$, $\vec{u}$ and $\vec{v}$

So arrow notation helps us see the algebra of vectors graphically.

Ex Let $\vec{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$. Sketch $\vec{u}, \vec{v}, \frac{1}{2}\vec{u}, -\vec{v}, \vec{u}+\vec{v}, \vec{u}-\vec{v}$

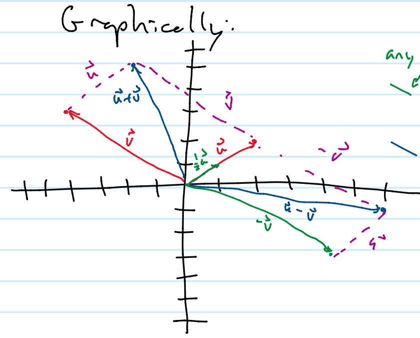
Algebraically:

* $u = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ sv

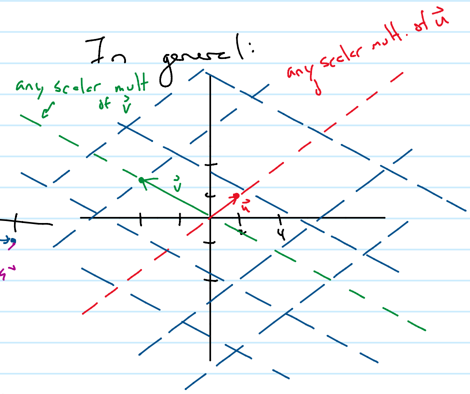
* $\frac{1}{2}\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $-\vec{v} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

* $\vec{u}+\vec{v} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, $\vec{u}-\vec{v} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$

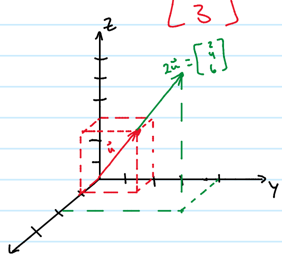
Graphically:



In general:



Ex Sketch $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $2\vec{u}$ in $\mathbb{R}^3$.



A couple of notes on linear combinations:

Possible linear combinations of $\vec{u}$ and $\vec{v}$

- the linear combinations of a single vector span the line it lies on.
- the linear combinations of two vectors with different directions (i.e. lying on different lines) span the plane containing those two lines

in $\mathbb{R}^2$ that's everything, in $\mathbb{R}^3$ and higher it isn't (just a subspace)

Span "where you can get with a set of vectors"

!! Fundamental

Def: For vectors $v_1, \dots, v_p$ in $\mathbb{R}^n$, the span of $v_1, \dots, v_p$ is the subset of $\mathbb{R}^n$ consisting of all linear combinations of $v_1, \dots, v_p$.

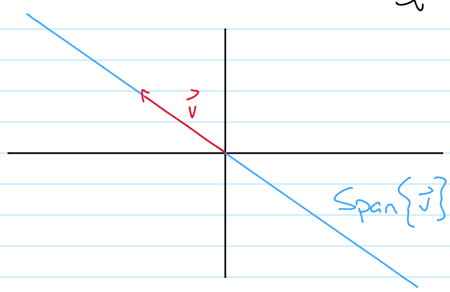
We denote this set $\text{Span}\{v_1, \dots, v_p\}$.

So, in other words $y \in$ ("is in") $\text{Span}\{v_1, \dots, v_p\}$ if there are $c_1, \dots, c_p \in \mathbb{R}$

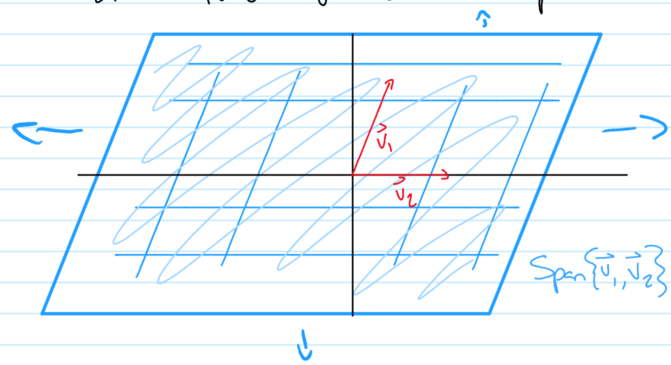
So, in other words $y \in \mathrm{Span}\{v_1, \dots, v_p\}$ ("is in") if there are $c_1, \dots, c_p \in \mathbb{R}$ s.t. $y = c_1\vec{v}_1 + \dots + c_p\vec{v}_p$.

Ex) In $\mathbb{R}^2$, Use online examples.

a one dimensional span

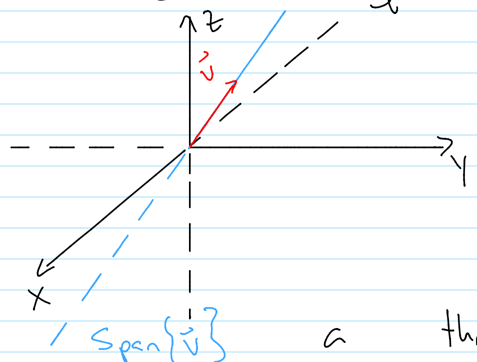


a two dimensional span.

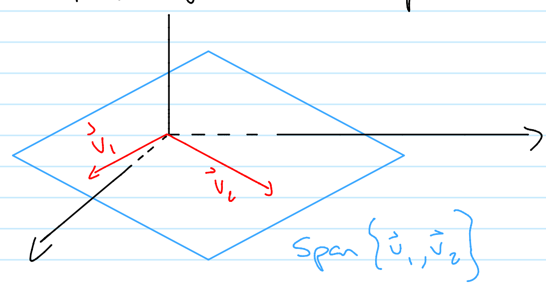


Ex) In $\mathbb{R}^3$

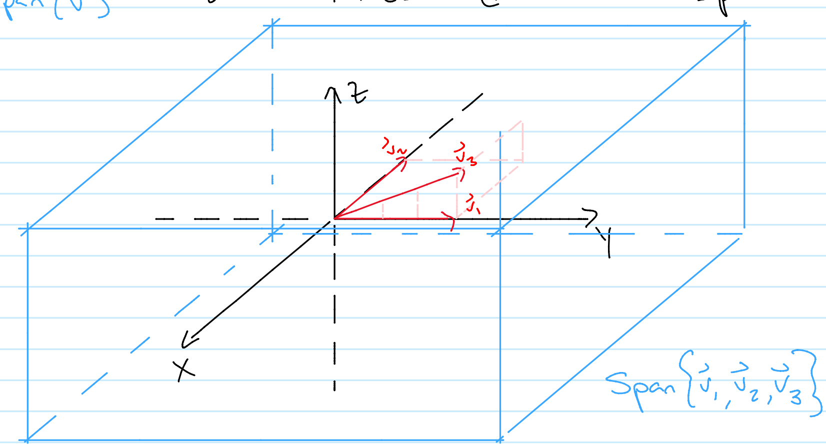
a one dimensional span



a two dimensional span.



a three dimensional span



Fundamental question: how do we determine when one vector is in the span of some collection of vectors?

!

In other words, if I have $v_1, \dots, v_p$, how do I know if some other $\vec{u}$ is a linear combination of those vectors?

That is, are there $c_1, \dots, c_p \in \mathbb{R}$ s.t.

$$\vec{u} = c_1\vec{v}_1 + \dots + c_p\vec{v}_p$$

or not?